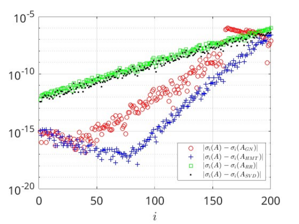
Given $\tilde{U} \in \mathbb{R}^{n\times r}, \tilde{V} \in \mathbb{R}^{n\times r}$ (orthonormal) **approximations** of the leading **singular vectors** of $A \in \mathbb{R}^{m\times n}$, $m \geq n$, **extract** the first r **singular values** $\{\sigma_i(A)\}_{i=1}^r$.

- Rayleigh Ritz: $\sigma_i(A) \approx \sigma_i(\tilde{U}^* A \tilde{V})$
- SVD approximation: $\sigma_i(A) \approx \sigma_i(A\tilde{V})$
- Generalized Nyström: $\sigma_i(A) \approx \sigma_i(A\tilde{V}(\tilde{U}^* A\tilde{V})^\dagger \tilde{U}^* A)$
- HMT: $A\tilde{V} = QR$, $\sigma_i(A) \approx \sigma_i(Q^* A)$



1. Interpret as Perturbation

$$Q_1^* \left(A - A_{GN,\tilde{V},\tilde{U}}\right) Q_2 = \begin{bmatrix} 0 & 0 \\ 0 & \bar{A}_{22} - \bar{A}_{21}\bar{A}_{11}^\dagger \bar{A}_{12} \end{bmatrix}$$

2. Matrix Perturbation Result

$$H := \begin{bmatrix} G_1 & B \\ C & G_2 \end{bmatrix} \quad F := \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}$$

Define, for $i = 1, \ldots, n$,

$$\tau_i = \left(\frac{\max\{\|B\|_2, \|C\|_2\} + \max\{\|F_{12}\|_2, \|F_{21}\|_2\}}{\min_k |\sigma_i - \sigma_k(G_2)| - 2\|F\|_2} \right).$$

Then, for all i for which $\tau_i > 0$, it holds

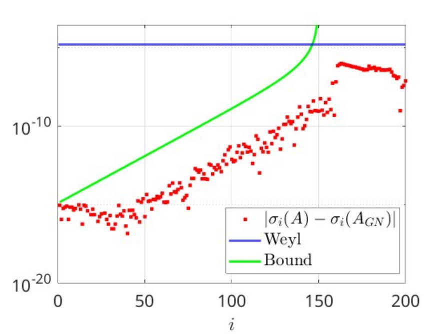
$$|\sigma_i - \hat{\sigma}_i| \leq \|F_{11}\|_2 + 2\max\{\|F_{12}\|_2, \|F_{21}\|_2\}\tau_i + \|F_{22}\|_2 \tau_i^2,$$

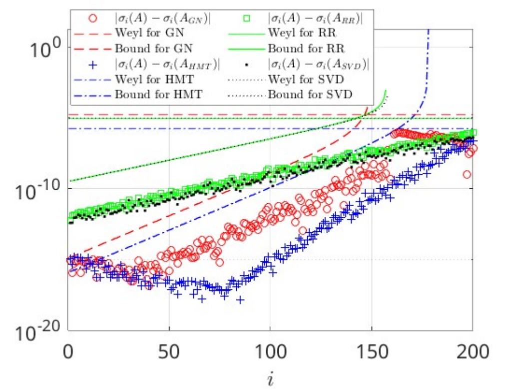
3. Application to Methods

$$\tau_i = \frac{\max\{\|\bar{A}_{12}\|_2, \|\bar{A}_{21}\|_2\}}{\min_k |\sigma_i - \sigma_k(\bar{A}_{22})| - 2\|E_{GN}\|_2}.$$

Then, for all i for which $\tau_i > 0$, it holds

$$|\sigma_i - \sigma_i^{GN}| \leq \left\|\bar{A}_{22} - \bar{A}_{21}\bar{A}_{11}^{\dagger}\bar{A}_{12}\right\|_2 \tau_i^2$$





- More formal comparison of methods
- Computability
- Oversampling vs No-oversampling?