ERROR BOUND ON SINGULAR VALUES APPROXIMATIONS BY GENERALIZED NYSTRÖM (GN)

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$$A = U \Sigma V^*$$

Given $\tilde{U}, \tilde{V}$ approximations of the leading singular vectors of A

$$\begin{array}{cc} & r \\ n & \begin{bmatrix} \\ \tilde{V} \\ \\ \end{bmatrix} \end{array}, \quad \begin{array}{cc} & r+\ell \\ m & \begin{bmatrix} \\ \\ \tilde{U} \\ \\ \\ \end{bmatrix} \end{array}$$

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$$\sigma_i(A) \approx \sigma_i(\tilde{U}^* A \tilde{V}) =: \sigma_i(A_{RR,\tilde{V},\tilde{U}})$$

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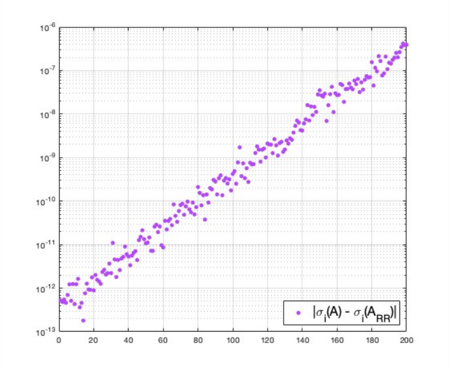

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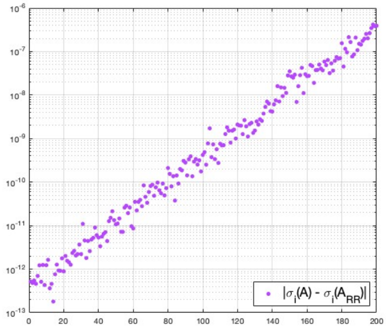

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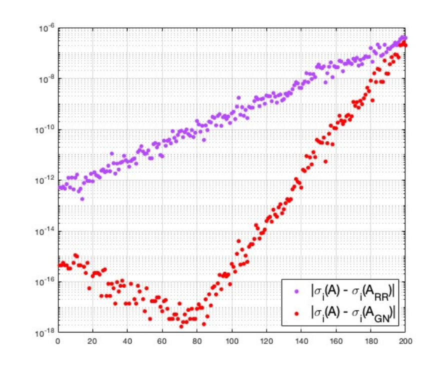

1. GN AS A PERTURBATION
2. MATRIX PERTURBATION THEORY RESULT
3. BOUND ON GN APPROXIMATION ERROR
4. FUTURE WORK

GN AS A PERTURBATION

GN and Orthogonal Transformations

Consider T_1 and T_2 orthogonal matrices, then

$$T_1^*(M_{GN,\tilde{V},\tilde{U}})T_2 = (T_1^* M T_2)_{GN,T_2^*\tilde{V},T_1^*\tilde{U}}$$

For any orthonormal $\tilde{V}$ and $\tilde{U}$, we can:

1. Define $Q_1 = \begin{bmatrix} \tilde{U} & \tilde{U}_\perp \end{bmatrix} \quad Q_2 = \begin{bmatrix} \tilde{V} & \tilde{V}_\perp \end{bmatrix}$;
2. Consider the transformed matrix: $Q_1^* A Q_2$;
3. Consider the transformed GN approximation:

$$Q_1^* A_{GN,\tilde{V},\tilde{U}} Q_2 = (Q_1^* A Q_2)_{GN,Q_2^*\tilde{V},Q_1^*\tilde{U}} = (Q_1^* A Q_2)_{GN,\begin{bmatrix} I_r \\ 0 \end{bmatrix},\begin{bmatrix} I_{r+\ell} \\ 0 \end{bmatrix}}.$$

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$$\rightarrow \quad |\sigma_i(A) - \sigma_i(A_{GN,\tilde{V},\tilde{U}})| = |\sigma_i(Q_1^* A Q_2) - \sigma_i((Q_1^* A Q_2)_{GN,\begin{bmatrix} I_r \\ 0 \end{bmatrix},\begin{bmatrix} I_{r+\ell} \\ 0 \end{bmatrix}})|$$

$$\tilde{V} := \begin{array}{c c} & r \\ \begin{array}{r} r \\ n-r \end{array} & \begin{bmatrix} I_r \\ - \\ 0 \end{bmatrix} \end{array}, \quad \tilde{U} := \begin{array}{c c} & r+\ell \\ \begin{array}{r} r+\ell \\ m-(r+\ell) \end{array} & \begin{bmatrix} I_{r+\ell} \\ - \\ 0 \end{bmatrix} \end{array}, \quad A := \begin{array}{c c} & \begin{array}{cc} r & n-r \end{array} \\ \begin{array}{r} r+\ell \\ m-(r+\ell) \end{array} & \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] \end{array}$$

Express A_{GN} as a perturbation of the original matrix A

$$A_{GN,\tilde{V},\tilde{U}} = A\tilde{V}\,(\tilde{U}^* A\tilde{V})^\dagger\,\tilde{U}^* A$$

$$\tilde{V} := \begin{array}{c} \\ r \\ n-r \end{array} \overset{\displaystyle r}{\begin{bmatrix} I_r \\ - \\ 0 \end{bmatrix}}, \quad \tilde{U} := \begin{array}{c} \\ r+\ell \\ m-(r+\ell) \end{array} \overset{\displaystyle r+\ell}{\begin{bmatrix} I_{r+\ell} \\ - \\ 0 \end{bmatrix}}, \quad A := \begin{array}{c} \\ r+\ell \\ m-(r+\ell) \end{array} \begin{array}{c} \begin{array}{cc} r & n-r \end{array} \\ \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] \end{array}$$

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Express A_{GN} as a perturbation of the original matrix A

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Express A_{GN} as a perturbation of the original matrix A

$$MM^\dagger M = M$$

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Express A_{GN} as a perturbation of the original matrix A

M has linearly independent columns
$\implies M^\dagger M = M^{-1} M = M$

$$A_{GN,\tilde{V},\tilde{U}} = \begin{bmatrix} A_{11} \\ - \\ A_{21} \end{bmatrix} (A_{11})^\dagger \left[\begin{array}{c|c} A_{11} & A_{12} \end{array}\right] = \left[\begin{array}{c|c} \overbrace{A_{11} A_{11}^\dagger A_{11}}^{= A_{11}} & A_{11} A_{11}^\dagger A_{12} \\ \hline A_{21} A_{11}^\dagger A_{11} & A_{21} A_{11}^\dagger A_{12} \end{array}\right]$$

$$\tilde V := \begin{array}{c c} & r \\ \begin{array}{c} r \\ n-r \end{array} & \begin{bmatrix} I_r \\ - \\ 0 \end{bmatrix} \end{array}, \quad \tilde U := \begin{array}{c c} & r+\ell \\ \begin{array}{c} r+\ell \\ m-(r+\ell) \end{array} & \begin{bmatrix} I_{r+\ell} \\ - \\ 0 \end{bmatrix} \end{array}, \quad A := \begin{array}{c c} & \begin{array}{cc} r & n-r \end{array} \\ \begin{array}{c} r+\ell \\ m-(r+\ell) \end{array} & \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] \end{array}$$

Express A_{GN} as a perturbation of the original matrix A

$$A_{GN,\tilde V,\tilde U} = \begin{bmatrix} A_{11} \\ - \\ A_{21} \end{bmatrix} (A_{11})^{\dagger} \left[\begin{array}{c|c} A_{11} & A_{12} \end{array}\right] = \left[\begin{array}{c|c} A_{11} & A_{11}A_{11}^{\dagger}A_{12} \\ \hline \underbrace{A_{21}A_{11}^{\dagger}A_{11}}_{=A_{21}} & A_{21}A_{11}^{\dagger}A_{12} \end{array}\right]$$

$$\tilde{V} := \begin{array}{c c} & r \\ \begin{array}{r} r \\ n-r \end{array} & \left[\begin{array}{c} I_r \\ \hline 0 \end{array}\right] \end{array}, \quad \tilde{U} := \begin{array}{c c} & r+\ell \\ \begin{array}{r} r+\ell \\ m-(r+\ell) \end{array} & \left[\begin{array}{c} I_{r+\ell} \\ \hline 0 \end{array}\right] \end{array}, \quad A := \begin{array}{c c} & \begin{array}{cc} r & n-r \end{array} \\ \begin{array}{r} r+\ell \\ m-(r+\ell) \end{array} & \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] \end{array}$$

Express A_{GN} as a perturbation of the original matrix A

$$A_{GN,\tilde{V},\tilde{U}} = A - \left[\begin{array}{c|c} 0 & A_{12} - A_{11}A_{11}^{\dagger}A_{12} \\ \hline 0 & A_{22} - A_{21}A_{11}^{\dagger}A_{12} \end{array}\right] =: A - E_{GN}$$

$$\tilde{V} := \begin{array}{c c} & r \\ r & \left[\begin{array}{c} I_r \\ \hline 0 \end{array}\right] \\ n-r & \end{array}, \quad \tilde{U} := \begin{array}{c c} & r+\ell \\ r+\ell & \left[\begin{array}{c} I_{r+\ell} \\ \hline 0 \end{array}\right] \\ m-(r+\ell) & \end{array}, \quad A := \begin{array}{c c} & \begin{array}{cc} r & n-r \end{array} \\ r+\ell & \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] \\ m-(r+\ell) & \end{array}$$

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Note: No-oversample ($\ell = 0$) $\rightarrow A_{12} - A_{11}A_{11}^{\dagger}A_{12} = 0$, <u>but</u> change of block sizes!

Weyl's Theorem

For any matrix M we have that

$$|\sigma_i(M) - \sigma_i(M+E)| \leq \|E\|_2$$

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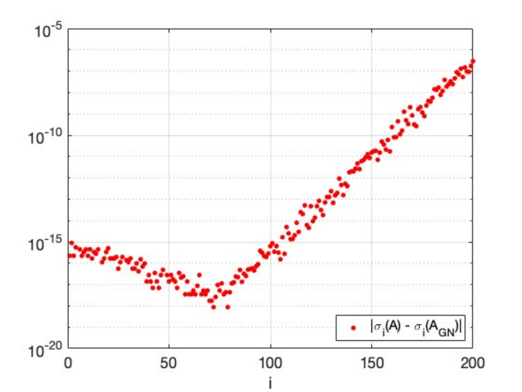
$$|\sigma_i(A) - \sigma_i(A_{GN,\tilde{V},\tilde{U}})|$$



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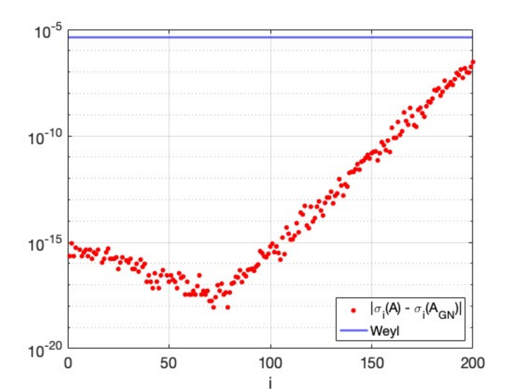

MATRIX PERTURBATION THEORY RESULT

Consider the $n \times n$ Hermitian matrices

$$H := \begin{bmatrix} H_{11} & H_{21}^* \\ H_{21} & H_{22} \end{bmatrix}, \quad \hat{H} := H + \begin{bmatrix} E_{11} & E_{21}^* \\ E_{21} & E_{22} \end{bmatrix} =: H + E.$$

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Theorem 3.2 (Nakatsukasa, 2012)

Define

$$\tau_i = \left(\frac{\|H_{21}\|_2 + \|E_{21}\|_2}{\min_j |\lambda_i(H) - \lambda_j(H_{22})| - 2\|E\|_2} \right).$$

Then, for each i, if $\tau_i > 0$, then

$$|\lambda_i(H) - \lambda_i(\hat{H})| \leq \|E_{11}\|_2 + 2\|E_{21}\|_2\tau_i + \|E_{22}\|_2\tau_i^2,$$

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- $\tau_i < 1$ necessary to be better than Weyl
- If $\|E_{11}\|_2 \ll \|E\|_2$ and λ_i is far from the spectrum of H_{22} then $\tau_i \ll 1$
- If $E_{11} = E_{21} = 0$ and H_{21} is small, then λ_i is particularly insensitive to the perturbation E_{22} $\rightarrow$ bound proportional to $\|E_{22}\|_2 \|H_{21}\|_2^2$

General case

$\downarrow$

Transform to Hermitian

$\downarrow$

Obtain necessary structure

$\downarrow$

Apply Hermitian Result

$\downarrow$

Transform back

$\downarrow$

General Result

Generalize (Nakatsukasa, 2012) to the non-Hermitian/rectangular case:

Consider the 2×2 block matrix:

$$G := \begin{bmatrix} G_1 & B \\ C & G_2 \end{bmatrix},$$

and its perturbation:

$$\hat{G} := G + \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} =: G + F.$$

Strategy: Use a technique in (Li, Li, 2005)

General case

$\downarrow$

Transform to Hermitian

$\downarrow$

Obtain necessary structure

$\downarrow$

Apply Hermitian Result

$\downarrow$

Transform back

$\downarrow$

General Result

Jordan-Wielandt (JW) Theorem

Let $\{\sigma_i(M)\}_{i=1}^n$ be the singular values of a matrix $M \in \mathbb{C}^{m \times n}$, with $m \geq n$. Then, the Hermitian matrix

$$\begin{bmatrix} 0 & M \\ M^* & 0 \end{bmatrix} \tag{1}$$

has eigenvalues $\pm\sigma_1(M), \ldots, \pm\sigma_n(M)$ and $m - n$ zeros eigenvalues.

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has eigenvalues $\pm\sigma_1(M), \ldots, \pm\sigma_n(M)$ and $m-n$ zeros eigenvalues.

$$G \to G_{JW} := \begin{bmatrix} 0 & | & G \\ - & - & - \\ G^* & | & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & | & G_1 & B \\ 0 & 0 & | & C & G_2 \\ - & - & - & - & - \\ G_1^* & C^* & | & 0 & 0 \\ B^* & G_2^* & | & 0 & 0 \end{bmatrix}$$

General case

↓

Transform to Hermitian

↓

Obtain necessary structure

↓

Apply Hermitian Result

↓

Transform back

↓

General Result

Obtain a matrix similar to G_{JW} suitable for (Nakatsukasa, 2012) and with blocks reasonably related to the blocks of G

$$\begin{bmatrix} 0 & 0 & | & G_1 & B \\ 0 & 0 & | & C & G_2 \\ - & - & - & - & - \\ G_1^* & C^* & | & 0 & 0 \\ B^* & G_2^* & | & 0 & 0 \end{bmatrix}$$

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$$\left[\begin{array}{cc|cc} 0 & 0 & G_1 & B \\ 0 & 0 & C & G_2 \\ \hline G_1^* & C^* & 0 & 0 \\ B^* & G_2^* & 0 & 0 \end{array}\right]$$

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$$\left[\begin{array}{cc|cc} 0 & G_1 & 0 & B \\ G_1^* & 0 & C^* & 0 \\ \hline 0 & C & 0 & G_2 \\ B^* & 0 & G_2^* & 0 \end{array}\right] =: G_p$$

Note: $\lambda_i(G_p) = \lambda_i(G_{JW}) \overset{JW}{=} \pm\sigma_i(G)$

General case

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General Result

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$$G_p = \begin{bmatrix} 0 & G_1 & | & 0 & B \\ G_1^* & 0 & | & C^* & 0 \\ - & - & - & - & - \\ 0 & C & | & 0 & G_2 \\ B^* & 0 & | & G_2^* & 0 \end{bmatrix}$$

$$\hat{G}_p = G_p + \begin{bmatrix} 0 & F_{11} & | & 0 & F_{12} \\ F_{11}^* & 0 & | & F_{21}^* & 0 \\ - & - & - & - & - \\ 0 & F_{21} & | & 0 & F_{22} \\ F_{12}^* & 0 & | & F_{22}^* & 0 \end{bmatrix} =: G_p + F_p.$$

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Transform to Hermitian

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General Result

Define

$$\tau_i = \left(\frac{\left\| \begin{bmatrix} 0 & C \\ B^* & 0 \end{bmatrix} \right\|_2 + \left\| \begin{bmatrix} 0 & F_{21} \\ F_{12}^* & 0 \end{bmatrix} \right\|_2}{\min_j \left| \lambda_i - \lambda_j \left(\begin{bmatrix} 0 & G_2 \\ G_2^* & 0 \end{bmatrix} \right) \right| - 2 \left\| F_p \right\|_2} \right).$$

Then, for each i, if $\tau_i > 0$:

$$|\lambda_i(G_p) - \lambda_i(\hat{G}_p)| \leq \left\| \begin{bmatrix} 0 & F_{11} \\ F_{11}^* & 0 \end{bmatrix} \right\|_2 + 2 \left\| \begin{bmatrix} 0 & F_{21} \\ F_{12}^* & 0 \end{bmatrix} \right\|_2 \tau_i + \left\| \begin{bmatrix} 0 & F_{22} \\ F_{22}^* & 0 \end{bmatrix} \right\|_2 \tau_i^2,$$

General case

$\downarrow$

Transform to Hermitian

$\downarrow$

Obtain necessary structure

$\downarrow$

Apply Hermitian Result

$\downarrow$

Transform back

$\downarrow$

General Result

- $\left\| \begin{bmatrix} 0 & M_1 \\ M_2 & 0 \end{bmatrix} \right\|_2 = \max\{\|M_1\|_2, \|M_2\|_2\};$

- Jordan-Wielandt theorem

$$\implies \quad |\lambda_i(G_p) - \lambda_i(\hat{G}_p)| = |\sigma_i(G) - \sigma_i(\hat{G})|,$$

  for $i = 1, \dots, n$;

- By Jordan-Wielandt theorem and by construction of F_p:

$$\|F_p\|_2 = \|F\|_2$$

General case

$\downarrow$

Transform to Hermitian

$\downarrow$

Obtain necessary structure

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Apply Hermitian Result

$\downarrow$

Transform back

$\downarrow$

General Result

Generalization of result in (Nakatsukasa, 2012)

Theorem (Al Daas, Lazzarino, Nakatsukasa)

Consider the matrices

$$G := \begin{bmatrix} G_1 & B \\ C & G_2 \end{bmatrix}, \quad \hat{G} := G + \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} =: G + F,$$

and define

$$\tau_i = \left(\frac{\max\{\|B\|_2, \|C\|_2\} + \max\{\|F_{12}\|_2, \|F_{21}\|_2\}}{\min_j |\sigma_i(G) - \sigma_j(G_2)| - 2\|F\|_2} \right).$$

Then, for each i, if $\tau_i > 0$, then

$$|\sigma_i(G) - \sigma_i(\hat{G})| \leq \|F_{11}\|_2 + 2\max\{\|F_{12}\|_2, \|F_{21}\|_2\}\tau_i + \|F_{22}\|_2\, \tau_i^2,$$

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- **Generalization to Block Tridiagonal**: A Singular Value is insensitive to blockwise perturbation if it is well-separated from the spectrum of the diagonal blocks near the perturbed blocks.

BOUND ON GN APPROXIMATION ERROR

Use previous results to obtain a bound on GN singular values approximation error

- $A, \tilde{V}, \tilde{U} \quad \rightarrow \quad A_{GN} = A\tilde{V}(\tilde{U}^* A\tilde{V})^\dagger \tilde{U}^* A$
- Define $\bar{A} = [\tilde{U}\ \tilde{U}_\perp]^* A[\tilde{V}\ \tilde{V}_\perp]$, $\quad \bar{A}_{GN} = \left([\tilde{U}\ \tilde{U}_\perp]^* A[\tilde{V}\ \tilde{V}_\perp]\right)_{GN, \begin{bmatrix} I_r \\ 0 \end{bmatrix}, \begin{bmatrix} I_{r+\ell} \\ 0 \end{bmatrix}}$

$$\implies \bar{A}_{GN} = \bar{A} - \begin{bmatrix} 0 & \bar{A}_{12} - \bar{A}_{11}\bar{A}_{11}^\dagger \bar{A}_{12} \\ 0 & \bar{A}_{22} - \bar{A}_{21}\bar{A}_{11}^\dagger \bar{A}_{12} \end{bmatrix} =: \bar{A} - E_{GN}$$

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Theorem, Bound on GN Singular Values Approximation Error (Al Daas, Lazzarino, Nakatsukasa)

Define

$$\tau_i = \frac{\max\{\|\bar{A}_{12}\|_2, \|\bar{A}_{21}\|_2\} + \left\|\bar{A}_{12} - \bar{A}_{11}\bar{A}_{11}^\dagger \bar{A}_{12}\right\|_2}{\min_j |\sigma_i(\bar{A}) - \sigma_j\left(\bar{A}_{22}\right)| - 2\|E_{GN}\|_2}.$$

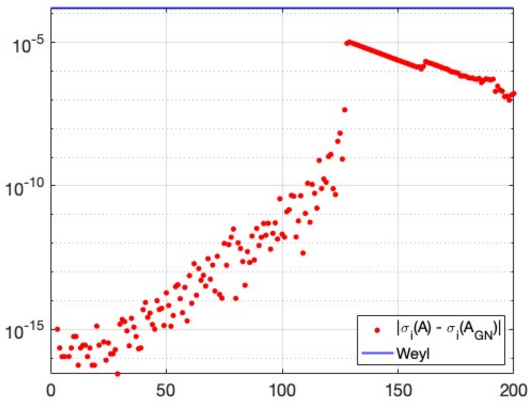
Then, for each i, if $\tau_i > 0$

$$|\sigma_i(A) - \sigma_i(A_{GN})| = |\sigma_i(\bar{A}) - \sigma_i(\bar{A}_{GN})| \leq 2\left\|\bar{A}_{12} - \bar{A}_{11}\bar{A}_{11}^\dagger \bar{A}_{12}\right\|_2 \tau_i + \left\|\bar{A}_{22} - \bar{A}_{21}\bar{A}_{11}^\dagger \bar{A}_{12}\right\|_2 \tau_i^2$$

- $\tau_i < 1$ necessary to be better than Weyl. If $\sigma_i(\bar{A})$ is far from the spectrum of $\bar{A}_{22}$ then $\tau_i \ll 1$

- $A \in \mathbb{R}^{1000 \times 1000}$
- U_{ex}, V_{ex} Haar Matrices
- $\sigma_i(A)$ exponentially decaying
- $[\tilde{V}, \sim] = \mathtt{qr}(A^*\Omega, 0)$
- $[\tilde{U}, \sim] = \mathtt{qr}(A\Omega, 0)$
- $\tilde{V} \in \mathbb{R}^{1000 \times 200}$
- $\tilde{U} \in \mathbb{R}^{1000 \times 200}$
- Compute pseudoinverses by QR factorization

$$\sigma_i(A_{GN,\tilde{V},\tilde{U}}) = \sigma_i(A\tilde{V}(\tilde{U}^*A\tilde{V})^\dagger \tilde{U}^*A)$$



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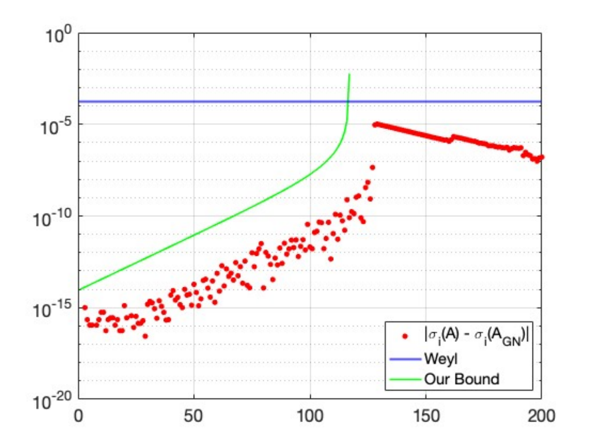

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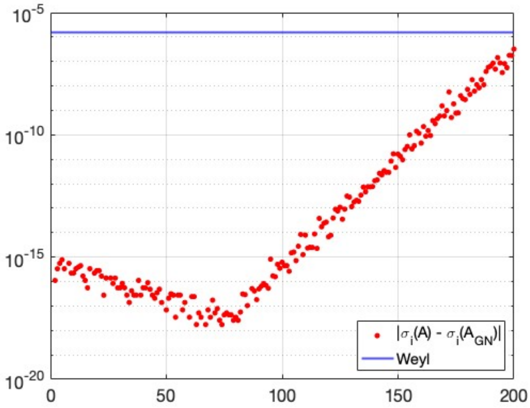

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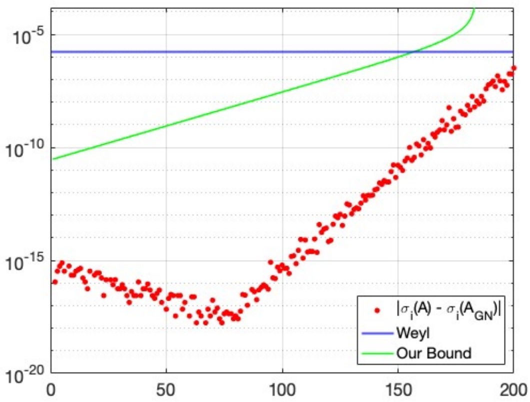
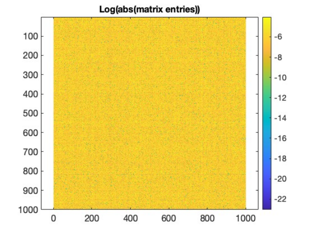
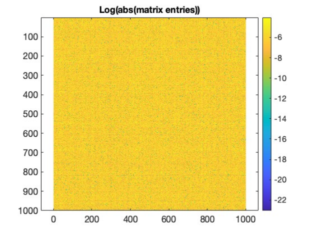
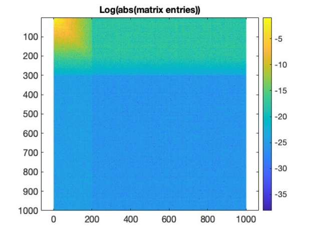


A


Log(abs(matrix entries))
100
200
300
400
500
600
700
800
900
1000
0
200
400
600
800
1000
-6
-8
-10
-12
-14
-16
-18
-20
-22

A $\rightarrow$ $\bar{A}$





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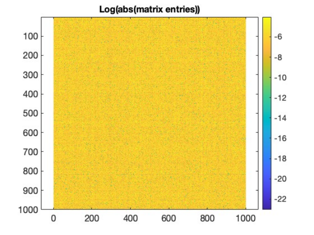
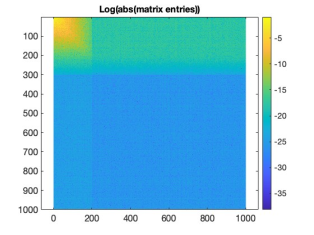
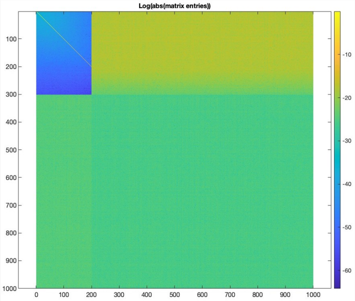


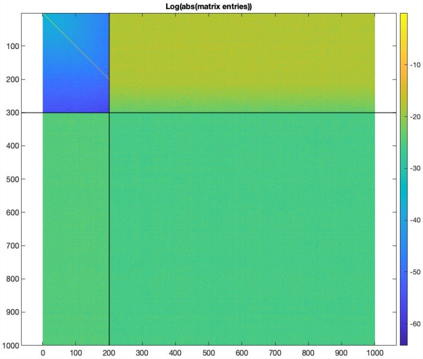


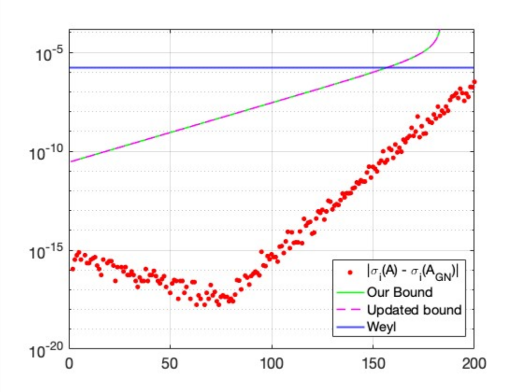
$\tilde{A}$



$$\tilde{V} \in \mathbb{R}^{1000 \times 200}$$
$$\tilde{U} \in \mathbb{R}^{1000 \times 300}$$

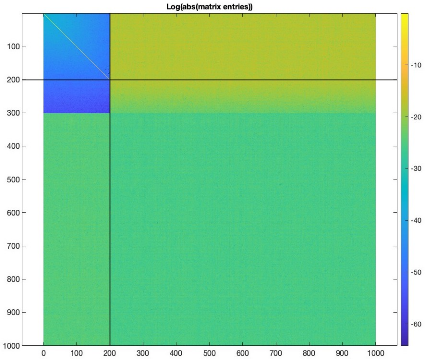
Size of $\tilde{A}_{11}$: 300×200

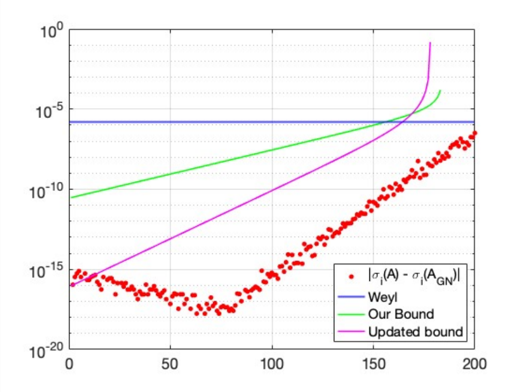




$$\tilde{V} \in \mathbb{R}^{1000 \times 200}$$
$$\tilde{U} \in \mathbb{R}^{1000 \times 300}$$

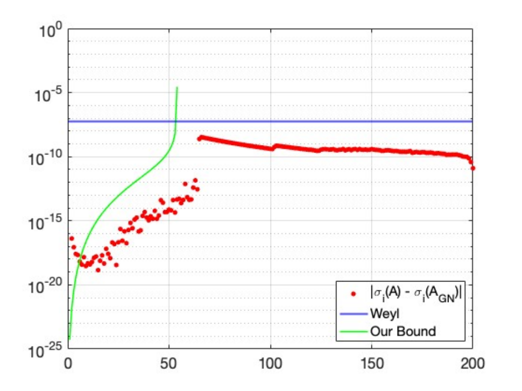
Size of $\tilde{A}_{11}$: 200×200



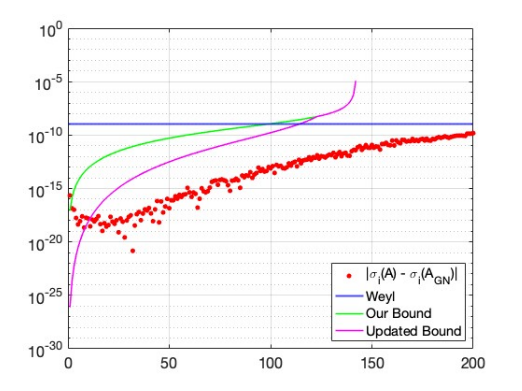


$$\sigma_i(A) = (\tfrac{1}{i})^4$$

Without oversample ($\ell = 0$)



With oversample ($r + \ell = 1.5r$)



Rayleigh Ritz (RR)

$$\sigma_i(A) \approx \sigma_i(\tilde{U}^* A \tilde{V}) =: \sigma_i(A_{RR,\tilde{V},\tilde{U}})$$

- $\bar{A} = [\tilde{U}\ \tilde{U}_\perp]^* A [\tilde{V}\ \tilde{V}_\perp]$
- $|\sigma_i(\bar{A}) - \sigma_i(\bar{A}_{RR, \begin{bmatrix} I_r \\ 0 \end{bmatrix}, \begin{bmatrix} I_{r+\ell} \\ 0 \end{bmatrix}})|$
- $\sigma_i(\bar{A}_{RR}) \overset{nnz}{=} \sigma_i \left(\bar{A} - \begin{bmatrix} 0 & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} \right)$

$\implies$ Perform the same analysis and derive a similar bound!

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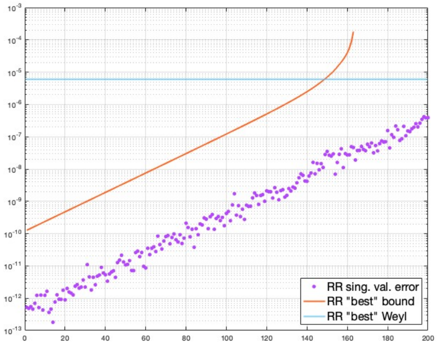

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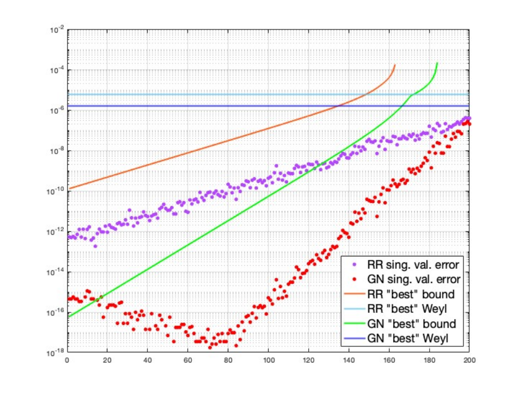

Provide ideas on how to make the bound computable in practice

For $\tau_i > 0, \quad |\sigma_i(A) - \sigma_i(A_{GN})| \leq 2 \left\| \bar{A}_{12} - \bar{A}_{11}\bar{A}_{11}^{\dagger}\bar{A}_{12} \right\|_2 \tau_i + \left\| \bar{A}_{22} - \bar{A}_{21}\bar{A}_{11}^{\dagger}\bar{A}_{12} \right\|_2 \tau_i^2$

$$\tau_i = \frac{\max\{\|\bar{A}_{12}\|_2, \|\bar{A}_{21}\|_2\} + \left\| \bar{A}_{12} - \bar{A}_{11}\bar{A}_{11}^{\dagger}\bar{A}_{12} \right\|_2}{\min_j |\sigma_i(\bar{A}) - \sigma_j\left(\bar{A}_{22}\right)| - 2\|E_{GN}\|_2}$$

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$$\text{(Backward Bound)} \quad \bar{A} = \bar{A}_{GN} + E_{GN} \implies \tau_i = \frac{\max\{\|\bar{A}_{11}\bar{A}_{11}^{\dagger}\bar{A}_{12}\|_2, \|\bar{A}_{12}\|_2\} + \|\bar{A}_{12} - \bar{A}_{11}\bar{A}_{11}^{\dagger}\bar{A}_{12}\|_2}{\min_j |\sigma_i(\bar{A}_{GN}) - \sigma_j(\bar{A}_{21}\bar{A}_{11}^{\dagger}\bar{A}_{12})| - 2\|E_{GN}\|_2}$$

Provide ideas on how to make the bound computable in practice

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$$\tau_i = \frac{\overbrace{\max\{\|\bar{A}_{11}\bar{A}_{11}^{\dagger}\bar{A}_{12}\|_2, \|\bar{A}_{12}\|_2\}}^{=\|\bar{A}_{12}\|_2} + \overbrace{\|\bar{A}_{12} - \bar{A}_{11}\bar{A}_{11}^{\dagger}\bar{A}_{12}\|_2}^{\leq\|\bar{A}_{12}\|_2}}{\min_j |\sigma_i(\bar{A}_{GN}) - \sigma_j(\bar{A}_{21}\bar{A}_{11}^{\dagger}\bar{A}_{12})| - 2\,\|E_{GN}\|_2}$$

Provide ideas on how to make the bound computable in practice

For $\tau_i > 0, \quad |\sigma_i(A) - \sigma_i(A_{GN})| \le 2 \left\| \bar{A}_{12} - \bar{A}_{11}\bar{A}_{11}^{\dagger}\bar{A}_{12} \right\|_2 \tau_i + \left\| \bar{A}_{22} - \bar{A}_{21}\bar{A}_{11}^{\dagger}\bar{A}_{12} \right\|_2 \tau_i^2$

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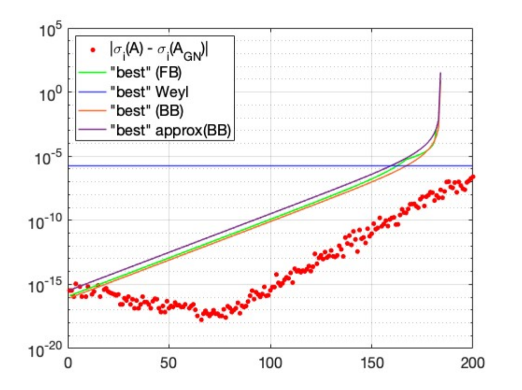

FUTURE WORK

- More on the difference between oversampled and non-oversampled cases
- More on the strategy to improve the bound;
- Use bounds to formally characterize the differences in behaviors of the different techniques: GN, HMT, Rayleigh-Ritz;
- Use norm estimation strategies to make the bound fully computable.

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Thank You!

$$\tilde{V} = \mathtt{qr}(A' * \mathtt{randn}(1000, 200));$$
$$\tilde{U} = \mathtt{qr}(A * \mathtt{randn}(1000, 300));$$

Size of $\tilde{A}_{11}$: 180×175